Chapter 11 Probabilistic Method

Definition A probability space is a triple $(\Omega, \Sigma, \mathrm{P})$, where Ω is a set, $\Sigma \subseteq 2^\Omega$ is a σ-algebra on Ω (a collection of subsets containing Ω and closed on complements, countable unions and countable intersections), and P is a countably additive measure on Σ with $\mathrm{P}[\Omega] = 1$. The elements of Σ are called events and the elements of Ω are called elementary events. For an event A, $\mathrm{P}[A]$ is called the probability of A.

We will consider Ω finite and $\Sigma = 2^\Omega$ in our examples later.

1: Give an example of $(\Omega, \Sigma, \mathrm{P})$.

ROLL A DIE $\quad \Omega = \{1,2,3,4,5,6\}$

Σ EVENTS $= 2^\Omega$

EVEN NUMBER $\{2,4,6\}$

EVENT $\{2\}$

$P[\{1\}] = \frac{1}{6}$

$P[\{i\}] = \frac{1}{6}$

EVEN Ω .. ANYTHING HAPPENED .. $P[\Omega] = 1$

$$P[A] = \sum_{a \in A} P[a]$$

2: Why is P on Σ and not on Ω?

PICK A RANDOM REAL NUMBER x FROM $[0,1]$

$P[x = \frac{1}{2}] = 0$

$P[x = \frac{1}{3}] = 0$

NOT TROUBLE FOR US

$P[x \in [0, 1/2]] = 1/2$

NOT ELEMENTARY EVENT, IN Σ

3: Show that for any collection of events $A_1, \ldots, A_n$, $\subseteq \Omega$

$$\mathrm{P}\left[\bigcup_{i=1}^n A_i\right] \leq \sum_{i=1}^n \mathrm{P}(A_i).$$

MAKE EVENTS

$B_i := A_i \setminus \bigcup_{j=1}^{i-1} A_j$

$P(A \cup B) = P(A) + P(B)$

IF $A \cap B = \emptyset$

$P(B_i) \leq P(A_i)$

$$P\left[\bigcup_{i=1}^n A_i\right] = P\left[\bigcup_{i=1}^n B_i\right] = \sum_{i=1}^n P[B_i] \leq \sum_{i=1}^n P[A_i]$$

Events A, B are independent if $\mathrm{P}[A \cap B] = \mathrm{P}[A]\mathrm{P}[B]$. More generally, events $A_1, \ldots, A_n$ are independent if for any subset of indices $I \subseteq [n]$

$$\mathrm{P}\left[\bigcap_{i \in I} A_i\right] = \prod_{i \in I} \mathrm{P}[A_i].$$

4: Find three events A_1, A_2 and A_3 that are pairwise independent but not mutually independent. (You need to say what is $(\Omega, \Sigma, \mathrm{P})$ as well.)
Hint: $\Omega = \{a, b, c, d\}$ *and* $\mathrm{P}[x] = \frac{1}{4}$ *for each* $x \in \Omega$ *could work.*

$A_1 = \{a, b\}$

$A_2 = \{a, c\}$

$A_3 = \{b, c\}$

$P[A_i \cap A_j] = \frac{1}{4}$ $\qquad P[A_i] = \frac{1}{2}$

$\frac{1}{4} = P[A_i] \cdot P[A_j]$

$P[A_1 \cap A_2 \cap A_3] = 0$ $\qquad \prod_{i=1}^{3} P[A_i] = \frac{1}{8}$

For events A and B with $\mathrm{P}[B] > 0$, we define the conditional probability of A, given that B occurs, as

$$\mathrm{P}[A|B] = \frac{\mathrm{P}[A \cap B]}{\mathrm{P}(B)}.$$

5: Simplify the formula for independent events A and B.

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A] \cdot P[B]}{P[B]} = P[A]$$

A real random variable on a probability space $(\Omega, \Sigma, \mathrm{P})$ is a function $X : \Omega \to \mathbb{R}$ that is P-measurable. (That is, for any $a \in \mathbb{B}, \{\omega \in \Omega : X(\omega) \leq a\} \in \Sigma$.)

We use Ω discrete, so no trouble with measurable in our case.

CHANCE HAPPENING

VALUE

"WEIGHTED AVERAGE"

Expectation for finite Ω can be expressed as $E[X] = \sum_{\omega\in\Omega} \mathrm{P}[\omega] X(\omega)$

Real random variables X, Y are independent if for every two measurable sets $A, B \subseteq \mathbb{R}$,

$$\mathrm{P}[X \in A \text{ and } Y \in B] = \mathrm{P}[X \in A] \cdot \mathrm{P}[Y \in B].$$

For verification, it is enough to check

$$\mathrm{P}[X \leq a \text{ and } Y \leq b] = \mathrm{P}[X \leq a] \cdot \mathrm{P}[Y \leq b].$$

6: What is $\mathrm{P}[X \in A]$?

$$P[\{\omega \in \Omega : X(\omega) \in A\}]$$

7: Show the following for a finite probability space. If X and Y are independent random variables, then $E[XY] = E[X] \cdot E[Y]$.

V_x V_y VALUES OF X AND Y (POSSIBLE VALUES)

$$E(XY) = \sum_{a\in V_x,\, b\in V_y} a \cdot b \; P[X = a \text{ AND } Y = b]$$

$$= \sum_{a\in V_x,\, b\in V_y} a \cdot b \; P[X = a] \cdot P[Y = b]$$

$$= \left(\sum_{a\in V_x} a \cdot P[X = a]\right) \cdot \left(\sum_{b\in V_y} b \cdot P[Y = b]\right)$$

$$= E(X) \cdot E(Y)$$

EACH EDGE HAS K VTXS

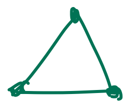
2-coloring hypergraphs - Construct something random.

A k-uniform hypergraph (V, E) has V as a set of vertices and edges $E \subseteq \binom{V}{k}$. That is, edges are k-subsets.

A hypergraph is c-colorable if its vertices can be colored with c colors so that no edge is monochromatic i.e., at least two different colors appear in every edge.

Let $m(k)$ denote the smallest number of edges in a k-uniform hypergraph that is not 2-colorable.

8: What is $m(2)$?



$m(2) = 3$

9: Use probabilistic method to show that for any $k \geq 2$,

$$m(k) \geq 2^{k-1}.$$

$m(2) \geq 2$

Hint: Union bound.

ASSUME $m < 2^{k-1}$ = # OF EDGES OF H -- K-UNIFORM HYPERGRAPH

COLOR VERTICES OF H RANDOMLY BY 2 COLORS

B_i := EVENT EDGE i IS MONOCHROMATIC $i \in [1, 2, .. m]$ GOAL $P[\cup B_i] < 1$

$$P[B_i] = \underbrace{2}_{\text{PICK COLOR}} \cdot \underbrace{\left(\frac{1}{2}\right)^k}_{\text{FOR EACH VTX}} = 2^{1-k} = \frac{2 \leftarrow \text{COUNTED}}{2^k \ \text{ALL}} = 2^{1-k}$$

$$P\left[\bigcup_i B_i\right] \leq \sum_i P[B_i] = m \cdot 2^{1-k} < 2^{k-1} \cdot 2^{1-k} = 1$$

Linearity of Expectation

Linearity of Expectation Let $X_1, \cdots, X_n$ be random variables, $X = c_1 X_1 + \cdots, +c_n X_n$, then

$$\mathbb{E}[X] = c_1 \mathbb{E}[X_1] + \cdots + c_n \mathbb{E}[X_n].$$

Definition For an event A, the indicator random variable I_A has value 1 if event A occurs and has value 0 otherwise.

10: Calculate the expected number of fixed points of random permuation σ on $\{1, \ldots, n\}$, i.e., the number of i such that $\sigma(i) = i$.

11: Show that there is a tournament on n vertices that has at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.
Remark: Alon(1990) proved that the maximum number of Hamiltonian paths is at most $cn^{3/2}\frac{n!}{2^{n-1}}$.

12: Show that any graph G with e edges contains a bipartite subgraph with at least $e/2$ edges.
Hint: randomly partition vertices into two parts.

The above result can be improved:

13: Show that if G has $2n$ vertices and e edges, then it contains a bipartite subgraph with at least $\frac{n}{2n-1}e$ edges.
If G has $2n+1$ vertices and e edges, then it contains a bipartite subgraph with at least $\frac{n+1}{2n+1}e$ edges

14: Given vectors $v_1, \ldots, v_n \in R^n$ with $|v_i| = 1$. Show that there exist $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$ such that

$$|\varepsilon_1 v_1 + \cdots + \varepsilon_n v_n| \leq \sqrt{n},$$

and also there exist $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$ such that

$$|\varepsilon_1 v_1 + \cdots + \varepsilon_n v_n| \geq \sqrt{n}.$$

Hint: pick ε_i randomly

15: Given vectors $v_1, \dots, v_n \in R^n$ with $|v_i| \leq 1$. Let $p_1, \dots, p_n \in [0,1]$ be arbitrary, and set $w = p_1 v_1 + \cdots + p_n v_n$. Then there exist $\varepsilon_1, \dots, \varepsilon_n \in \{0,1\}$ so that set $v = \varepsilon_1 v_1 + \cdots + \varepsilon_n v_n$, we have

$$|w - v| \leq \frac{\sqrt{n}}{2}.$$

16: Let F be a family of subsets of $[n] = \{1, \dots, n\}$ such that there are no $A, B \in F$ satisfying $A \subset B$. Let σ be a random permutation of $[n]$. Consider the random variable $X = |\{i : \{\sigma(1), \sigma(2), \dots, \sigma(i)\} \in F\}|$. Prove $|F| \leq \binom{n}{\lfloor n/2 \rfloor}$ by considering the expectation of X.

Some estimates:

$$n! \le n^n \qquad n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\left(\frac{n}{e}\right)^n \le n! \le en\left(\frac{n}{e}\right)^n \qquad \left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

$$\frac{2^{2m}}{\sqrt{2m}} \le \binom{2m}{m} \le \frac{2^{2m}}{2\sqrt{m}}$$

$$(1-p)^m \le e^{-pm} \qquad (1-p) \ge e^{-2p} \text{ for } 0 \le p \le \frac{1}{2}$$

17: *(Bonus)* Let $(\Omega, 2^\Omega, P)$ be a finite probability space, where all elementary events have the same probability. Show that if $|\Omega|$ is a prime, then there does not exist a pair of non-trivial independent events. Trivial events are $\emptyset$ and Ω.